

Analysis of a retrial queue with two phase service and server vacations

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Abstract

A queueing system with a single server providing two stages of service in succession is considered. Every customer receives service in the first stage and in the sequel he decides either to proceed to the second phase of service or to depart and to join a retrial box from where he repeats the demand for a special second stage service after a random amount of time and independently of the other customers in the retrial box. When the server becomes idle he departs for a single vacation of an arbitrarily distributed length. The arrival process is assumed to be Poisson and all service times are arbitrarily distributed. For such a system the stability conditions and the system state probabilities are investigated both in a transient and in a steady state. Numerical results are finally obtained and used to investigate system performance.

Keywords: Poisson arrivals, two-phase service, retrial queue, general services, single vacation.

1 Introduction

Queueing systems in which the server provides to each customer two phases of heterogeneous service in succession, have been proved very useful to model computer networks, production lines and telecommunication systems where messages are processed in two stages by a single server. Such kind of systems have firstly discussed by Krishna and Lee [10] and Doshi [6], while more recently, in a series of works of Madan [13], Choi and Kim [2], Choudhury and Madan [4], Katayama and Kobayashi [9], the previous results are extended to include models allowing server vacations, Bernoulli feedback, N-policy, exhaustive or gated bulk service etc. Moreover in all papers mentioned above one can find important applications of the two phase service models to computer communication, production and manufacturing systems, central processor and multimedia communications.

Kumar, Vijayakumar and Arivudainambi [12] and Choudhury [3] are the first who imposed the concept of "retrial customers" in the two phase service models. Retrial queueing systems are characterized by the fact that an arriving customer who finds the server unavailable does not wait in a queue but instead he leaves the system joining the so called retrial box from where he repeats the demand for service later. Practical use of retrial queueing systems arises in telephone-switching systems and in telecommunication and computer networks. For complete surveys of past papers on such kind of models see Falin and Templeton [7], Kulkarni and Liang [11] and Artalejo [1]. Kumar et.al. [12] considered a two phase service system where an arriving customer who finds the server unavailable joins the retrial box from where only the first customer can retry for service after an arbitrarily distributed time period while in the work of Choudhury [3] the investigated model includes Bernoulli server vacations and linear retrial policy. We have to observe here that in both papers there is not any ordinary queue and all "waiting" customers are placed in the retrial box.

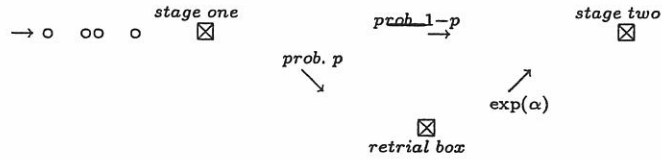
In the work here we consider a two phase service model where now all arriving customers are waiting in an ordinary queue to receive service in the first stage. When a customer completes his first phase service then with probability $1 - p$ he proceeds to the second phase while with probability p he leaves the system and joins the retrial box from where he retries, after a random amount of time, to find the server available and to complete a special second phase of service. Moreover, when there are not on-line customers for service, the server departs for a single vacation (update devices, maintenance, etc) of arbitrarily distributed length. Our system can be used to model any situation with two stages of service where in the first stage a control and a separation of the serviced units according to some quality standards or some measure of importance must be taken place. If a unit satisfies these quality standards then it proceeds immediately to the second phase of service while if the quality of the unit is poor then it is removed from the system and repeat its attempt to receive a special second service later when the server is free from high quality units. As one understand a such kind of situation arise often in packet transmissions, in manufacturing systems, in central processors, in multimedia communications etc. Note here that in our model and at any time, an ordinary and a retrial queue must be taken in to account and so the analysis become much more complicated.

The article is organized as follows. A full description of the model is given in section 2. The time dependent analysis of the system state probabilities is performed in section 3 while some, very useful for the analysis, results on the customer completion time and server busy period are given in section 4. In section 5 the conditions for statistical equilibrium are investigated. The generating functions of the steady state probabilities are obtained in section 6 and used to give, in section 7, some important measures of the system performance. Numerical results are obtained finally in section 8 and used to compare system performance under various changes of the parameters.

2 The model

Consider a queueing system consisting of two phases of service and a single server who follows the customer in service when he passes from the first phase to the second. Customers arrive to the system according to a *Poisson* distribution parameter λ , and are placed in a single queue waiting to be served. When a customer finishes his service in the first phase then either he goes to the second phase with probability $1 - p$, or he departs from the system with probability p and joins a retrial box from where he retries, independently to the other customers in the box, after an exponential time parameter α , to find the server idle and to complete a special second phase of service. In case the customer chooses to depart and to join the retrial box the server starts immediately to serve in the first phase the next customer in queue (if any). Every time the server becomes idle (no customers waiting in the ordinary queue) he departs for a single vacation U_0 which length is arbitrarily distributed with distribution function (D.F.) $B_0(x)$, probability density function (p.d.f.) $b_0(x)$ and finite mean value \bar{b}_0 and second moment about zero $\bar{b}_0^{(2)}$.

Let us call P_1 customers the ordinary customers who are queued up and wait to be served and P_2 customers those who join the retrial box. The service times in both phases are assumed to be arbitrarily distributed with D.F. $B_{ij}(x)$, p.d.f. $b_{ij}(x)$ and finite mean value \bar{b}_{ij} and second moment about zero $\bar{b}_{ij}^{(2)}$ for the P_i customer in the j^{th} phase respectively $i, j = 1, 2$, ($B_{21}(x)$, $b_{21}(x)$, \bar{b}_{21} , $\bar{b}_{21}^{(2)}$ do not exist). Finally all random variables defined above are assumed to be independent. The following figure explains the situation.



3 System states analysis

Let $N_i(t)$, $i = 1, 2$, be the P_i customers in the system at time t and denote

$$\xi_t = \begin{cases} 0 & \text{server on vacation at } t, \\ (i, j) & \text{server busy on } j \text{ phase with } P_i \text{ customer at } t, \\ id & \text{server idle at } t, \end{cases}$$

and

$$\begin{aligned} q(k_2, t) &= P(N_1(t) = 0, N_2(t) = k_2, \xi_t = id), \\ p_0(k_1, k_2, x, t) dx &= P(N_1(t) = k_1, N_2(t) = k_2, \xi_t = 0, x < \bar{U}_0(t) \leq x + dx), \\ p_{ij}(k_1, k_2, x, t) dx &= P(N_1(t) = k_1, N_2(t) = k_2, \xi_t = (i, j), x < \bar{U}_{ij}(t) \leq x + dx), \end{aligned}$$

where $\bar{U}_{ij}(t)$, $\bar{U}_0(t)$ are the elapsed service or vacation time respectively. If finally

$$\begin{aligned} Q(z_2, t) &= \sum_{k_2 \geq 0} q(k_2, t) z_2^{k_2}, \\ P_0(z_1, z_2, x, t) &= \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} p_0(k_1, k_2, x, t) z_1^{k_1} z_2^{k_2}, \\ P_{ij}(z_1, z_2, x, t) &= \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} p_{ij}(k_1, k_2, x, t) z_1^{k_1} z_2^{k_2}, \quad i, j = 1, 2, \end{aligned}$$

and denote by $Q^*(z_2, s)$, $P_0^*(z_1, z_2, x, s)$, $P_{ij}^*(z_1, z_2, x, s)$ the corresponding LS-transforms, then by connecting as usual t and $t + dt$ we arrive easily, for $x > 0$, at

$$\begin{aligned} P_0^*(z_1, z_2, x, s) &= P_0^*(z_1, z_2, 0, s)(1 - B_0(x)) \exp[-(s + \lambda - \lambda z_1)x], \\ P_{ij}^*(z_1, z_2, x, s) &= P_{ij}^*(z_1, z_2, 0, s)(1 - B_{ij}(x)) \exp[-(s + \lambda - \lambda z_1)x], \end{aligned} \quad (1)$$

and

$$\alpha z_2 \frac{d}{dz_2} Q^*(z_2, s) + (\lambda + s) Q^*(z_2, s) = 1 + P_0^*(0, z_2, 0, s) \beta_0^*(s + \lambda), \quad (2)$$

with $\beta_0^*(\cdot)$, $\beta_{ij}^*(\cdot)$ the LS transforms of $b_0(\cdot)$, $b_{ij}(\cdot)$ respectively. For the boundary conditions ($x = 0$) we obtain in a similar way

$$\begin{aligned} P_0^*(0, z_2, 0, s) &= p z_2 P_{11}^*(0, z_2, 0, s) \beta_{11}^*(s + \lambda) + P_{12}^*(0, z_2, 0, s) \beta_{12}^*(s + \lambda) \\ &\quad + P_{22}^*(0, z_2, 0, s) \beta_{22}^*(s + \lambda), \\ P_{12}^*(z_1, z_2, 0, s) &= (1 - p) P_{11}^*(z_1, z_2, 0, s) \beta_{11}^*(s + \lambda - \lambda z_1), \\ P_{22}^*(0, z_2, 0, s) &= \alpha \frac{d}{dz_2} Q^*(z_2, s), \end{aligned} \quad (3)$$

while

$$\begin{aligned} z_1 P_{11}^*(z_1, z_2, 0, s) &= p z_2 (P_{11}^*(z_1, z_2, 0, s) \beta_{11}^*(s + \lambda - \lambda z_1) - P_{11}^*(0, z_2, 0, s) \beta_{11}^*(s + \lambda)) \\ &\quad + P_0^*(0, z_2, 0, s) (\beta_0^*(s + \lambda - \lambda z_1) - \beta_0^*(s + \lambda)) + P_{12}^*(z_1, z_2, 0, s) \\ &\quad \times \beta_{12}^*(s + \lambda - \lambda z_1) - P_{12}^*(0, z_2, 0, s) \beta_{12}^*(s + \lambda) + P_{22}^*(z_1, z_2, 0, s) \\ &\quad \times \beta_{22}^*(s + \lambda - \lambda z_1) - P_{22}^*(0, z_2, 0, s) \beta_{22}^*(s + \lambda) + \lambda z_1 Q^*(z_2, s). \end{aligned} \quad (4)$$

Substituting finally from (1), (3) to (4) we arrive at

$$P_{11}^*(z_1, z_2, 0, s) = \frac{\lambda z_1 Q^*(z_2, s) + \alpha \beta_{22}^*(s + \lambda - \lambda z_1) \frac{d}{dz_2} Q^*(z_2, s) - P_0^*(0, z_2, 0, s) [1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda z_1)]}{z_1 - \beta_{11}^*(s + \lambda - \lambda z_1) [p z_2 + (1 - p) \beta_{12}^*(s + \lambda - \lambda z_1)]}. \quad (5)$$

To proceed further we need the following Lemma the proof of which is a simple application of the well known theorem of Takacs [15].

Lemma 1 For (i) $|z_2| < 1$, $\text{Re}(s) \geq 0$, or (ii) $|z_2| \leq 1$, $\text{Re}(s) > 0$, or (iii) $|z_2| \leq 1$, $\text{Re}(s) \geq 0$ and $\rho_1 = \lambda(\bar{b}_{11} + (1 - p)\bar{b}_{12}) > 1$, the relation

$$z_1 - \beta_{11}^*(s + \lambda - \lambda z_1) [p z_2 + (1 - p) \beta_{12}^*(s + \lambda - \lambda z_1)], \quad (6)$$

has one and only one root, $z_1 = x(s, z_2)$ say, inside the region $|z_1| < 1$. Specifically for $s = 0$ and $z_2 = 1$, $x(0, 1)$ is the smallest positive real root of (6) with $x(0, 1) < 1$ if $\rho_1 > 1$ and $x(0, 1) = 1$ for $\rho_1 \leq 1$.

Replacing now the zero of the denominator in the numerator in (5) we arrive at

$$P_0^*(0, z_2, 0, s) = \frac{\lambda x(s, z_2)Q^*(z_2, s) + \alpha\beta_{22}^*(s + \lambda - \lambda x(s, z_2))\frac{d}{dz_2}Q^*(z_2, s)}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))}, \quad (7)$$

and substituting in (2) we obtain

$$\alpha(z_2 - D_2(s, z_2))\frac{d}{dz_2}Q^*(z_2, s) + (s + \lambda - \lambda D_1(s, z_2))Q^*(z_2, s) = 1, \quad (8)$$

with

$$\begin{aligned} D_2(s, z_2) &= \frac{\beta_0^*(s + \lambda)\beta_{22}^*(s + \lambda - \lambda x(s, z_2))}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))}, \\ D_1(s, z_2) &= \frac{x(s, z_2)\beta_0^*(s + \lambda)}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))}. \end{aligned} \quad (9)$$

Let us define now

$$\rho = \rho_1 + \lambda p \bar{b}_{22} + \frac{\lambda p \bar{b}_0}{\beta_0^*(\lambda)}. \quad (10)$$

Then one can show the following

Theorem 2 For (i) $\text{Re}(s) > 0$, or (ii) $\text{Re}(s) \geq 0$ and $\rho > 1$, the equation

$$z_2 - D_2(s, z_2) = 0, \quad (11)$$

has one and only one root, $z_2 = \phi(s)$ say, inside the region $|z_2| < 1$. Specifically for $s = 0$, $\phi(0)$ is the smallest positive real root of (11) with $\phi(0) < 1$ if $\rho > 1$ and $\phi(0) = 1$ for $\rho \leq 1$.

Proof: The proof of the theorem is completely based on the concept of the "generalized completion time of a retrial customer" that will be investigated in section 4. Thus comparing $D_2(s, z_2)$ in the first of (9) with the function $w_2^*(s, z_2)$ in (16) of section 4 one realizes easily that

$$D_2(s, z_2) \equiv w_2^*(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^\infty w_m^{(2)}(t) z_2^m dt,$$

where $w_m^{(2)}(t)$ is a probability density, i.e. $D_2(s, z_2)$ is in fact the Laplace transform of a generating function.

Thus for the closed contour $|z_2| = 1$ and under the assumption (i) we have always

$$|D_2(s, z_2)| \leq D_2(\text{Re}(s), 1) < D_2(0, 1) = 1 \equiv |z_2|,$$

while for $\text{Re}(s) \geq 0$, we need to consider the closed contour $|z_2| = 1 - \epsilon$ ($\epsilon > 0$ a small number) in which case

$$|D_2(s, z_2)| \leq D_2(\text{Re}(s), 1 - \epsilon) < 1 - \epsilon \equiv |z_2|, \quad (12)$$

only if in addition

$$\frac{d}{d\epsilon} D_2(0, 1 - \epsilon) |_{\epsilon=0} = -\frac{1}{1 - \rho_1} [\lambda p \bar{b}_{22} + \frac{\lambda p \bar{b}_0}{\beta_0^*(\lambda)}] < \frac{d}{d\epsilon} (1 - \epsilon) |_{\epsilon=0} = -1,$$

or we need $\rho > 1$ for the relation (12) to hold. A final reference to Rouché's theorem completes the first part of the proof.

Moreover for $s = 0$ the convex function $D_2(0, z_2)$ is a monotonically increasing function of z_2 , for $0 \leq z_2 \leq 1$, taking the values $D_2(0, 0) < 1$ and $D_2(0, 1) = 1$ and so $0 < \phi(0) < 1$ if $\rho > 1$, while for $\rho \leq 1$, $\phi(0)$ becomes equal to 1 and this completes the proof. \square

Using the theorem above one can solve (see Falin and Fricker [8]) the differential equation (8) and obtain

$$Q^*(s, z_2) = \frac{1}{s + \lambda - \lambda D_1(s, z_2)}, \quad \text{if } z_2 = \phi(s),$$

$$Q^*(s, z_2) = \int_{z_2}^{\phi(s)} \frac{1}{a(D_2(s, u) - u)} \exp\left\{ \int_u^{z_2} \frac{s + \lambda - \lambda D_1(s, x)}{a(D_2(s, x) - x)} dx \right\} du, \quad \text{if } z_2 \neq \phi(s).$$

Thus the quantity $Q^*(s, z_2)$ is known and so from the (1), (3), (5) and (7) all generating functions are completely known. This completes the time-dependent analysis of the model.

4 Generalized completion time and busy period

Let us define

$$\begin{aligned} B^{(i)} &= \text{Duration of a Busy Period of } P_1 \text{ customers starting} \\ &\quad \text{with } i \text{ } P_1 \text{ customers,} \\ N(B^{(i)}) &= \text{New } P_2 \text{ customers joining the retrial box during } B^{(i)}, \\ g_m^{(i)}(t) dt &= P(t < B^{(i)} \leq t + dt, N(B^{(i)}) = m), \end{aligned}$$

then it is clear that

$$g^{(i)}(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^\infty g_m^{(i)}(t) z_2^m dt = x^{(i)}(s, z_2),$$

where $x(s, z_2)$ is defined in the Lemma 1 above.

Let now C be the random interval from the epoch a P_2 (retrial) customer finds a position for service until the epoch the server departs for the single vacation, and let $N(C)$ be the number of the new customers joining the retrial box during C . If we define $c_m(t) dt = P(t < C \leq t + dt, N(C) = m)$ then

$$c_0(t) = e^{-\lambda t} b_{22}(t) + \sum_{i=1}^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} b_{22}(t) * g_0^{(i)}(t),$$

$$c_m(t) = \sum_{i=1}^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} b_{22}(t) * g_m^{(i)}(t),$$

where $*$ means convolution, and after manipulations we obtain

$$c^*(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^\infty c_m(t) z_2^m dt = \beta_{22}^*(s + \lambda - \lambda x(s, z_2)). \quad (13)$$

In a similar way if we denote by V the random interval from the epoch the server departs for a vacation until the epoch he is for the first time idle, and by $N(V)$ the number of the new customers joining the retrial box during V and define

$$v_m(t) dt = P(t < V \leq t + dt, N(V) = m),$$

$$v^*(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^\infty v_m(t) z_2^m dt,$$

then

$$v_0(t) = e^{-\lambda t} b_0(t) + \sum_{i=1}^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} b_0(t) * g_0^{(i)}(t) * v_0(t),$$

$$v_m(t) = \sum_{i=1}^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} b_0(t) * \sum_{k=0}^m g_k^{(i)}(t) * v_{m-k}(t),$$

and so

$$v^*(s, z_2) = \frac{\beta_0^*(s + \lambda)}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))}. \quad (14)$$

Now we are ready to define the concepts of the Generalized Completion time and Generalized busy period. **Generalized completion time**, W_2 say, of a P_2 (retrial) customer is the time elapsed from the instant that this customer finds a position for service until the instant the server is idle for the first time, while **generalized busy period**, W_1 say, is the time interval from the epoch a P_1 customer arrives in an idle system until the epoch the server is again idle. If now we denote by $N(W_2)$, $N(W_1)$ the number of new retrial customers joining the retrial box in W_2 , W_1 , respectively, and define

$$w_m^{(i)}(t) dt = P(t < W_i \leq t + dt, N(W_i) = m), \quad i = 1, 2,$$

$$w_i^*(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^\infty w_m^{(i)}(t) z_2^m dt,$$

then it is clear that

$$w_2^*(s, z_2) = c^*(s, z_2) v^*(s, z_2), \quad w_1^*(s, z_2) = x^*(s, z_2) v^*(s, z_2), \quad (15)$$

and so

$$w_2^*(s, z_2) = \frac{\beta_0^*(s + \lambda) \beta_{22}^*(s + \lambda - \lambda x(s, z_2))}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))},$$

$$w_1^*(s, z_2) = \frac{x(s, z_2) \beta_0^*(s + \lambda)}{1 + \beta_0^*(s + \lambda) - \beta_0^*(s + \lambda - \lambda x(s, z_2))}. \quad (16)$$

By differentiating finally with respect to z_2 to the point $(z_2 = 1, s = 0)$ the obtained in this section relations we arrive easily at

$$\frac{d}{dz_2} x^*(0, z_2)|_{z_2=1} = \frac{p}{1 - \rho_1}, \quad \rho_1 = \lambda(\bar{b}_{11} + (1 - p)\bar{b}_{12})$$

$$\frac{d}{dz_2} c^*(0, z_2)|_{z_2=1} = \frac{\rho_2}{1 - \rho_1}, \quad \rho_2 = \lambda p \bar{b}_{22}, \quad (17)$$

$$\frac{d}{dz_2} v^*(0, z_2)|_{z_2=1} = \frac{\rho_0}{1 - \rho_1}, \quad \rho_0 = \frac{\lambda p \bar{b}_0}{\beta_0^*(\lambda)}, \quad (18)$$

and so

$$E(N(W_2)) = \frac{d}{dz_2} w_2^*(0, z_2)|_{z_2=1} = \frac{\rho_0 + \rho_2}{1 - \rho_1}, \quad (19)$$

$$E(N(W_1)) = \frac{d}{dz_2} w_1^*(0, z_2)|_{z_2=1} = \frac{p + \rho_0}{1 - \rho_1}. \quad (20)$$

Moreover by differentiating with respect to s to the point $(z_2 = 1, s = 0)$ we arrive at

$$\begin{aligned} E(W_1) &= \frac{p\rho_1 + \rho_0}{\lambda p(1 - \rho_1)}, \\ E(W_2) &= \frac{\rho_2 + \rho_0}{\lambda p(1 - \rho_1)}. \end{aligned} \quad (21)$$

5 Stability Conditions

Consider the time instants

$$T_0 = 0 < T_1 < T_2 < \dots,$$

where T_i is the epoch at which the server becomes idle for the i^{th} time, and let $N_{2i} = N_2(T_i + 0)$, $i = 0, 1, 2, \dots$, i.e. N_{2i} denote the number of customers in the retrial box just after T_i . It is clear that the stochastic process $\{N_{2i} : i = 0, 1, 2, \dots\}$ is an irreducible and aperiodic Markov chain. The following theorem gives the condition under which this Markov chain becomes positive recurrent. Note that using (17) and (18) the quantity ρ defined in (10) can be written

$$\rho = \rho_0 + \rho_1 + \rho_2.$$

Theorem 3 *For $\rho < 1$ the Markov chain $\{N_{2i} : i = 0, 1, 2, \dots\}$ is positive recurrent.*

Proof: To prove the theorem, we will use the following criterion (see Pakes [14]):

An irreducible and aperiodic Markov chain $(Y_n ; n \geq 0)$, with state space the nonnegative integers, is positive recurrent if $|\delta_k| < \infty$ for all $k = 0, 1, 2, \dots$ and $\limsup_{k \rightarrow \infty} \delta_k < 0$, where $\delta_k = E[Y_{n+1} - Y_n | Y_n = k]$.

For the Markov chain of our model, let

$$h_{k,m}(t)dt = \Pr[t < T_{n+1} - T_n \leq t + dt, N_{2n+1} - N_{2n} = m | N_{2n} = k].$$

Then it is easy to see that for $m = 0, 1, 2, \dots$

$$h_{k,m}(t) = \lambda e^{-(\lambda+ka)t} * w_m^{(1)}(t) + kae^{-(\lambda+ka)t} * w_{m+1}^{(2)}(t),$$

while for $m = -1$

$$h_{k,-1}(t) = kae^{-(\lambda+ka)t} * e^{-\lambda t} b_{22}(t) * v_0(t) \\ + kae^{-(\lambda+ka)t} * \sum_{i=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!} b_{22}(t) * g_0^{(i)}(t) * v_0(t),$$

and so

$$\int_0^{\infty} e^{-st} \sum_{m=-1}^{\infty} h_{k,m}(t) z^m dt = \frac{\lambda w_1^*(s, z) + \frac{ka}{z} w_2^*(s, z)}{s + \lambda + ka}. \quad (22)$$

By taking derivatives above with respect to z at the point $(z = 1, s = 0)$ we arrive at

$$\delta_k = \frac{\lambda E(N(W_1)) + ka[E(N(W_2)) - 1]}{\lambda + ka}, \quad k = 0, 1, \dots,$$

where $E(N(W_1))$, $E(N(W_2))$ have been found in (20) and (19) respectively.

Thus for $\rho < 1$ we realize that $|\delta_k|$ is finite for all k and also $\limsup_{k \rightarrow \infty} \delta_k = E(N(W_2)) - 1 = \frac{\rho_0 + \rho_2}{1 - \rho_1} - 1 < 0$, and the criterion is satisfied. \square

For a stochastic process $(Y(t); t \geq 0)$ we will say that it is stable, if its limiting probabilities as $t \rightarrow \infty$ exist and form a distribution. Consider now the stochastic process

$$\mathbf{Z} = \{(N_1(t), N_2(t), \xi_t) : 0 \leq t < \infty\},$$

where $N_i(t)$, ξ_t have been defined in section 3. Then

Theorem 4 For $\rho < 1$ the process \mathbf{Z} is stable.

Proof: Consider the quantity

$$m_k = E(T_1 | N_{20} = k).$$

By taking derivatives in (22) with respect to s (at $z = 1, s = 0$) we obtain

$$m_k = \frac{\lambda E(W_1) + kaE(W_2) + 1}{\lambda + ka},$$

and if q_k $k = 0, 1, 2, \dots$, are the steady state probabilities of the positive recurrent (for $\rho < 1$) Markov chain $\{N_{2i} : i = 0, 1, 2, \dots\}$ then

$$\mathbf{q} \cdot \mathbf{m} = \sum_{k=0}^{\infty} q_k m_k = E(W_2) + \{1 + \lambda[E(W_1) - E(W_2)]\} \sum_{k=0}^{\infty} \frac{q_k}{\lambda + ka}. \quad (23)$$

Now it is clear that there is always a finite integer k^* such that

$$\frac{1}{\lambda + (k^* - 1)a} > 1 > \frac{1}{\lambda + k^*a},$$

and so

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{q_k}{\lambda + ka} &= \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda + ka} + \sum_{k=k^*}^{\infty} \frac{q_k}{\lambda + ka} < \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda + ka} \\ &+ \sum_{k=k^*}^{\infty} q_k = \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda + ka} + (1 - \sum_{k=0}^{k^*-1} q_k) < \infty, \end{aligned}$$

and so from (23) using (21) we understand that $\mathbf{q} \cdot \mathbf{m} < \infty$.

Consider finally the irreducible aperiodic and positive recurrent Markov Renewal Process $\{N, T\} = \{(N_{2n}, T_n) : n = 0, 1, 2, \dots\}$. It is easy to see that the stochastic process \mathbf{Z} is a Semi-Regenerative Process with imbedded Markov Renewal Process $\{N, T\}$ and as (for $\rho < 1$) $\mathbf{q} \cdot \mathbf{m} < \infty$ it is clear that \mathbf{Z} is, for $\rho < 1$, stable (Cinlar [5], Theorem 6.12 p.347). \square

6 Steady State Probabilities

Suppose now that $\rho < 1$ and define

$$\begin{aligned} p_0(k_1, k_2, x) &= \lim_{t \rightarrow \infty} p_0(k_1, k_2, x, t), \\ q(k_2) &= \lim_{t \rightarrow \infty} q(k_2, t), \\ p_{ij}(k_1, k_2, x) &= \lim_{t \rightarrow \infty} p_{ij}(k_1, k_2, x, t), \end{aligned}$$

$$\begin{aligned} Q(z_2) &= \sum_{k_2 \geq 0} q(k_2) z_2^{k_2}, \\ P_0(z_1, z_2, x) &= \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} p_0(k_1, k_2, x) z_1^{k_1} z_2^{k_2}, \\ P_{ij}(z_1, z_2, x) &= \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} p_{ij}(k_1, k_2, x) z_1^{k_1} z_2^{k_2}, \quad i, j = 1, 2. \end{aligned}$$

Then it is well known that $P_0(z_1, z_2, x) = \lim_{s \rightarrow 0} s P_0^*(z_1, z_2, x, s)$, $Q(z_2) = \lim_{s \rightarrow 0} s Q^*(z_2, s)$, and so from (1) and (2) we obtain

$$\begin{aligned} P_0(z_1, z_2, x) &= P_0(z_1, z_2, 0)(1 - B_0(x)) \exp[-(\lambda - \lambda z_1)x], \\ P_{ij}(z_1, z_2, x) &= P_{ij}(z_1, z_2, 0)(1 - B_{ij}(x)) \exp[-(\lambda - \lambda z_1)x], \end{aligned} \quad (24)$$

and

$$\alpha z_2 \frac{d}{dz_2} Q(z_2) + \lambda Q(z_2) = P_0(0, z_2, 0) \beta_0^*(\lambda), \quad (25)$$

while the boundary conditions (3) and (4) become

$$\begin{aligned} P_0(0, z_2, 0) &= p z_2 P_{11}(0, z_2, 0) \beta_{11}^*(\lambda) + P_{12}(0, z_2, 0) \beta_{12}^*(\lambda) + P_{22}(0, z_2, 0) \beta_{22}^*(\lambda), \\ P_{12}(z_1, z_2, 0) &= (1 - p) P_{11}(z_1, z_2, 0) \beta_{11}^*(\lambda - \lambda z_1), \\ P_{22}(0, z_2, 0) &= \alpha \frac{d}{dz_2} Q(z_2), \end{aligned} \quad (26)$$

and

$$P_{11}(z_1, z_2, 0) = \frac{\lambda z_1 Q(z_2) + \alpha \beta_{22}^*(\lambda - \lambda z_1) \frac{d}{dz_2} Q(z_2) - P_0(0, z_2, 0) [1 + \beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda z_1)]}{z_1 - \beta_{11}^*(\lambda - \lambda z_1) [p z_2 + (1 - p) \beta_{12}^*(\lambda - \lambda z_1)]}. \quad (27)$$

Replacing now in the numerator of (27) the zero (in $|z_1| < 1$) $x(z_2) \equiv x(0, z_2)$ of the denominator we obtain

$$P_0(0, z_2, 0) = \frac{\lambda x(z_2)Q(z_2) + \alpha\beta_{22}^*(\lambda - \lambda x(z_2))\frac{d}{dz_2}Q(z_2)}{1 + \beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda x(z_2))}. \quad (28)$$

Substituting (28) to (25) we arrive easily at

$$\alpha(z_2 - w_2^*(0, z_2))\frac{d}{dz_2}Q(z_2) + \lambda(1 - w_1^*(0, z_2))Q(z_2) = 0. \quad (29)$$

Let now

$$\omega(z_2) = \frac{1 - w_1^*(0, z_2)}{z_2 - w_2^*(0, z_2)},$$

then for $\rho < 1$ the quantity $z_2 - w_2^*(0, z_2)$ never becomes zero in $|z_2| < 1$ (Theorem 2) and also

$$\lim_{z_2 \rightarrow 1} \omega(z_2) = -\frac{\rho + \rho_0}{1 - \rho} < \infty.$$

Thus $\omega(z_2)$ is an analytic function in $|z_2| < 1$ and a continuous one on the boundary and so for any $|z_2| \leq 1$ we can solve equation (29) and obtain

$$Q(z_2) = Q(1) \exp\left\{-\frac{\lambda}{a} \int_{z_2}^1 \frac{1 - w_1^*(0, u)}{w_2^*(0, u) - u} du\right\}.$$

Replacing finally $Q(z_2)$ back in the generating functions and asking for the total probabilities to sum to unity we arrive at

$$Q(1) = \frac{1 - \rho}{1 + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}},$$

and so the generating functions of the steady state probabilities are completely known.

The following theorem shows that the condition $\rho < 1$ is also necessary for a stable system.

Theorem 5 *If the stochastic process \mathbf{Z} is stable then $\rho < 1$.*

Proof: Suppose that \mathbf{Z} is stable and $\rho > 1$. Then from theorem 2 the equation $z_2 - w_2^*(0, z_2) = 0$ has a root strictly less than one ($\phi(0) < 1$) and so $\lambda(1 - w_1^*(0, \phi(0))) \neq 0$. By putting now $\phi(0)$ instead of z_2 in (29) we obtain

$$\lambda(1 - w_1^*(0, \phi(0)))Q(\phi(0)) = 0,$$

and so $Q(\phi(0)) = \sum q(j)\phi^j(0) = 0$ with $0 < \phi(0) < 1$. Thus $q(j) = 0 \ \forall j$ and also from the generating functions in (24)-(28) it is clear that all probabilities become zero. This of course contradicts to the hypothesis that the system is stable.

Suppose finally that \mathbf{Z} is stable and $\rho = 1$. By taking derivatives with respect to z_2 in (29) (at $z_2 = 1$) we arrive (for $\rho = 1$) at

$$\frac{d}{dz_2} \lambda(1 - w_1^*(0, z_2))|_{z_2=1} Q(1) = -\lambda E(N(W_1)) Q(1) = 0,$$

and so $Q(1) = \sum q(j) = 0$ and this again contradicts to the hypothesis that the system is stable.

7 Performance measures

In this section we will use formulas for the generating functions obtained previously, to derive expressions for the probabilities of some important measures of system performance. Thus by putting $z_1 = z_2 = 1$ into relations (24)-(28) we obtain easily

$$\begin{aligned} P[\text{server Idle}] &= Q(1) = \frac{1-\rho}{1+\frac{\lambda b_0}{\beta_0^*(\lambda)}}, \\ P[a P_1 \text{ customer in service in stage 1}] &= P_{11}(1, 1) = \lambda \bar{b}_{11}, \\ P[a P_1 \text{ customer in service in stage 2}] &= P_{12}(1, 1) = \lambda(1-p)\bar{b}_{12}, \\ P[a P_2 \text{ customer in service in stage 2}] &= P_{22}(1, 1) = \lambda p \bar{b}_{22}, \\ P[\text{server in vacation}] &= P_0(1, 1) = \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} (p + \frac{1-\rho}{1+\frac{\lambda b_0}{\beta_0^*(\lambda)}}). \end{aligned}$$

To obtain now the mean number of customers in the ordinary queue and in the retrial box we have to differentiate relations (24)-(28) with respect to z_1 and z_2 respectively at the point $(z_1, z_2) = (1, 1)$. Thus after manipulations

$$\begin{aligned} E(N_1, \xi = (1, 1)) &= \frac{\lambda^3 \bar{b}_{11}}{2(1-\rho_1)} \rho^{(2)} + \frac{\lambda^2 \bar{b}_{11}^{(2)}}{2}, \\ E(N_1, \xi = (1, 2)) &= \frac{\lambda^3 (1-p) \bar{b}_{12}}{2(1-\rho_1)} \rho^{(2)} + \lambda^2 (1-p) (\bar{b}_{11} \bar{b}_{12} + \frac{\bar{b}_{12}^{(2)}}{2}), \\ E(N_1, \xi = (2, 2)) &= \frac{\lambda^2 p \bar{b}_{22}^{(2)}}{2}, \\ E(N_1, \xi = 0) &= \frac{\lambda^2 \bar{b}_0^{(2)}}{2\beta_0^*(\lambda)} (p + \frac{1-\rho}{1+\frac{\lambda b_0}{\beta_0^*(\lambda)}}), \end{aligned} \tag{30}$$

and

$$\begin{aligned} E(N_2, \xi = (1, 1)) &= K \bar{b}_{11}, \\ E(N_2, \xi = (1, 2)) &= K(1-p) \bar{b}_{12}, \\ E(N_2, \xi = (2, 2)) &= \bar{b}_{22} D, \\ E(N_2, \xi = 0) &= \frac{\bar{b}_0}{\beta_0^*(\lambda)} (\frac{\lambda p(\lambda+\alpha)}{\alpha} + D), \\ E(N_2, \xi = id) &= \frac{\lambda p}{\alpha}, \end{aligned}$$

where

$$D = \frac{\lambda p}{1-\rho} \left\{ \frac{\lambda}{\alpha} (p + \rho_0) + \frac{\lambda^2 p \rho^{(2)}}{2(1-\rho_1)} + \lambda p \bar{b}_{11} + \rho_0 \right\},$$

$$\rho^{(2)} = \bar{b}_{11}^{(2)} + (1-p)\bar{b}_{12}^{(2)} + 2(1-p)\bar{b}_{11}\bar{b}_{12} + p\bar{b}_{22}^{(2)} + \frac{p\bar{b}_0^{(2)}}{\beta_0^*(\lambda)} \left(1 + \frac{1-\rho}{p+\rho_0}\right),$$

$$K = \frac{\lambda^3 p \rho^{(2)}}{2(1-\rho)(1-\rho_1)} + \frac{\lambda^2(p+\rho_0)}{\alpha(1-\rho)} + \frac{\lambda(\lambda p \bar{b}_{11} + \rho_0)}{1-\rho}.$$

We have to observe here that if we send, in our model, the mean retrial rate $1/\alpha$ to zero then all customers in the retrial box seem to retry continuously for service and so they become in fact ordinary customers but now of lower priority compared with the P_1 customers. Thus now the model becomes a system of two parallel queues. All customers receive a first phase of service in the first queue and in the sequel some of them (with probability p) join a second queue of lower priority while the remaining customers proceed immediately to the second phase of service. It is clear that in a such kind of model the server starts serving in this second queue in a non preemptive basis only when there are no more customers in the first queue.

Thus, assuming $\alpha \rightarrow \infty$ in the formulae obtained above, we arrive after elementary manipulations at the following results for the mean number of customers in the lower priority queue of this modified non-retrial model. Note that the number of customers in the first (high priority) queue does not affected from variations in α and so formulae (30) hold unaltered for the mean number of P_1 customers in the modified model.

$$\begin{aligned} E(N_2, \xi = (1, 1)) &= Z\bar{b}_{11}, \\ E(N_2, \xi = (1, 2)) &= Z(1-p)\bar{b}_{12}, \\ E(N_2, \xi = (2, 2)) &= \frac{\rho_2}{1-\rho} \left(\frac{\lambda^2 p \rho^{(2)}}{2(1-\rho_1)} + \rho_0 + \lambda p \bar{b}_{11} \right), \\ E(N_2, \xi = 0) &= \frac{\rho_0}{1-\rho} \left(1 - \rho + \frac{\lambda^2 p \rho^{(2)}}{2(1-\rho_1)} + \rho_0 + \lambda p \bar{b}_{11} \right), \end{aligned}$$

where

$$Z = \frac{\lambda^3 p \rho^{(2)}}{2(1-\rho)(1-\rho_1)} + \lambda \left(\frac{\rho_0 + \lambda p \bar{b}_{11}}{1-\rho} \right).$$

8 Numerical results

In this section we use the formulae derived previously to obtain numerical results and to investigate the way the mean number of customers in the retrial box $E(N_2)$ is affected when we vary the values of the parameters.

To construct the tables we assumed that the vacation time U_0 and the service times follow exponential distributions with p.d.f.'s respectively,

$$b_0(x) = \frac{1}{\bar{b}_0} e^{-(1/\bar{b}_0)x}, \quad b_{ij}(x) = \frac{1}{\bar{b}_{ij}} e^{-(1/\bar{b}_{ij})x}, \quad i, j = 1, 2.$$

Moreover we assume that in all tables below $\bar{b}_{12} = 0.5$, $p = 0.5$.

Table 1 shows the way $E(N_2)$ changes when we vary the mean vacation time \bar{b}_0 for increasing values of the mean arrival rate λ . Here one can observe that even for a small value of λ , $\lambda = 0.2$ for example, $E(N_2)$ increases from 0.229

to 66.323 when we pass from a system without vacation period ($\bar{b}_0 = 0$) to the system with $\bar{b}_0 = 3.9$. When now the arrival rate λ increases to $\lambda = 0.6$ then even a small change from $\bar{b}_0 = 0$ to $\bar{b}_0 = 0.4$ increases dramatically the mean number of retrial customers to 1696.1. Thus we must be very careful on the vacation period that we must allow, to avoid a rather overloaded retrial box.

λ/\bar{b}_0	0	0.2	0.4	0.6	0.8	1	1.3	2.2	3.9
0.2	0.229	0.2779	0.3353	0.4025	0.4818	0.5755	0.7503	1.718	66.323
0.3	0.5131	0.641	0.8083	1.0287	1.3243	1.7299	2.6926	54.231	
0.4	1.1378	1.4866	2.0188	2.8701	4.3641	7.4778	38.347		
0.45	1.7538	2.3771	3.4485	5.5148	10.682	41.261			
0.5	2.8366	4.0762	6.6607	14.125	140.2				
0.55	4.9698	7.9557	17.347	1248.7					
0.6	10.037	20.887	1696.1						
0.65	27.659	435.68							
0.7	427.81								

Table 1 : Values of $E(N_2)$ for $\alpha = 0.8$, $\bar{b}_{11} = 1$, $\bar{b}_{22} = 0.33$

Table 2 contains values of $E(N_2)$ when we vary the mean retrial rate $E(\text{retrial}) = 1/\alpha$. The first column ($E(\text{retrial}) = 0$) corresponds to the non-retrial model (second queue with low priority customers). One can observe here the effect of the retrial rate on the number of retrial customers and also one can make conclusions on the mean retrial interval that must allowed to achieve a suitably small size of the retrial box.

$\lambda/E(\text{retrial})$	0	0.02	0.2	1	2	10
0.2	0.0908	0.0938	0.1207	0.2405	0.3902	1.5877
0.3	0.2739	0.2797	0.3264	0.5676	0.8613	3.2109
0.4	0.7905	0.8016	0.9019	1.3474	1.9043	6.3598
0.45	1.3918	1.4076	1.5495	2.18	2.9683	9.2742
0.5	2.6105	2.6339	2.845	3.783	4.9556	14.336
0.55	5.5417	5.5803	5.9279	7.4729	9.4041	24.854
0.6	15.749	15.831	16.571	19.859	23.969	56.851
0.65	354.17	355.48	367.22	419.38	484.58	1006.2

Table 2 : Values of $E(N_2)$ for $\bar{b}_0 = 0.2$, $\bar{b}_{11} = 1$, $\bar{b}_{22} = 0.33$

Table 3 and Table 4 contain values of $E(N_2)$ when we vary the mean first stage service \bar{b}_{11} and the mean second stage service of retrial customers \bar{b}_{22} respectively. One can observe again the way the mean number of retrial customers $E(N_2)$ increases when \bar{b}_{11} or \bar{b}_{22} increases, particularly for large values of λ .

λ/\bar{b}_{11}	0.7	0.8	1	1.1	1.4	1.6	1.9	2.8	4.4
0.2	0.2295	0.2465	0.2779	0.2987	0.3797	0.4551	0.6129	1.892	403.3
0.3	0.4585	0.5083	0.641	0.7295	1.1397	1.6206	3.0542	1700.5	
0.4	0.8615	1.0148	1.4866	1.8565	4.2892	9.4264	101.65		
0.45	1.1843	1.4536	2.3771	3.2012	11.155	58.268			
0.5	1.6503	2.1352	4.0762	6.1912	67.333				
0.6	3.4947	5.3823	20.887	104.29					
0.65	5.5195	10.073	435.68						
0.75	21.845	1852.1							
0.8	151.65								

Table 3 : Values of $E(N_2)$ for $\bar{b}_0 = 0.2, \alpha = 0.8, \bar{b}_{22} = 0.33$

λ/\bar{b}_{22}	0.2	0.6	0.9	1.2	1.7	2.2	3.9	7.2
0.2	0.2697	0.2962	0.3202	0.3487	0.409	0.4915	1.1084	105.56
0.3	0.6072	0.721	0.8369	0.991	1.3794	2.07	125.1	
0.4	1.356	1.8293	2.4178	3.405	7.6509	66.169		
0.45	2.105	3.1561	4.7433	8.3593	207.52			
0.5	3.4511	6.1607	12.349	67.354				
0.55	6.2201	16.151	404.98					
0.6	13.469	678.52						
0.65	48.304							

Table 4 : Values of $E(N_2)$ for $\bar{b}_0 = 0.2, \alpha = 0.8, \bar{b}_{11} = 1$

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